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Unit-3.

Limits and Continuity

Def:- Modulus function:-

The function $f: R \to R$ defined by $f(x) = |x|$.
and it is defined $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
is called Modulous. function.

Def:- Integral partfunction:-

Let The function $f: R \to R$ be defined by $f(x) = [x]$.
is called an integral part of x.

Def:- Limit of a function:-

The function f from limit 'l' as x tends to 'a' if,
"To each $\varepsilon > 0$, $\exists\ \delta > 0 \ni$, $|f(x) - l| < \varepsilon$ whenever
$0 < |x - a| < \delta$. we write $f(x) \to l$ as $x \to a$."
It is denoted by $\lim_{x \to a} f(x) = l$.

i) Right hand limit:- The function f tends to limit l as $x \to a$ from right. if, to each $\varepsilon > 0$,
$\exists\ \delta > 0$, $\ni$ $a < x < a + \delta \Rightarrow |f(x) - l| < \varepsilon$.
we write $f(x) \to l$ as $x \to a$ (or) $\lim_{x \to a^+} f(x) = l$ (or) $f(a+0) = l$

ii) Left hand limit:- The function f tends to limit l as $x \to a$ from left. if to each $\varepsilon > 0$,
$\exists\ \delta > 0$, $\ni$ $a - \delta < x < a \Rightarrow |f(x) - l| < \varepsilon$ we write
$f(x) \to l$ (or) $f(a-0) = l$.

Hence the two Conditions are Satisfies then 'f' is called

Limit of a function.

Ex:- L.H.L $= f(a^-) = \lim_{x \to a^-} f(x)$

put $x = a-h$

Then $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

R.H.L $= f(a^+) = \lim_{x \to a^+} f(x)$

put $x = a+h$

Then $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$.

Prob:-

① Proove that $\lim_{x \to 0} \frac{3x+|x|}{7x-5|x|}$ does not exist.

Sol:- Given $f(x) = \frac{3x+|x|}{7x-5|x|}$, $|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$

Now i) L.H.L. $= \lim_{\substack{x \to a^- \\ x<a}} f(x) = \lim_{x \to 0^-} \frac{3x-x}{7x-5(-x)}$

$= \lim_{x \to 0^-} \frac{2\cancel{x}}{12\cancel{x}}$

$= \frac{1}{6}$

ii) R.H.L $= \lim_{\substack{x \to a^+ \\ x>a}} f(x) = \lim_{x \to 0^+} \frac{3x+x}{7x-5x}$

$= \lim_{x \to 0^+} \frac{4\cancel{x}}{2\cancel{x}}$

$= 2$

$\therefore$ L.H.L $\neq$ R.H.L.

Hence. $\lim_{x \to 0} \frac{3x + |x|}{7x - 5|x|}$ does not exist.

2) Prob:- Let $f: R \to R$ is a function defined $f(x) = \frac{|x-2|}{x-2}$ when $x \neq 2$. and $f(x) = 0$ when $x = 2$ prove that $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist.

Sol: Given $f(x) = \frac{|x-2|}{x-2}$.

where $|x-2| = \begin{cases} x-2 & \text{if } x-2 > 0 \Rightarrow x > 2 \\ -(x-2) & \text{if } x-2 < 0 \Rightarrow x < 2 \end{cases}$

Now i) L.H.L :- $\lim_{\substack{x \to 2^- \\ x < 2}} f(x) = \lim_{x \to 2^-} \frac{|x-2|}{x-2}$

$= \lim_{x \to 2^-} \frac{-\cancel{(x-2)}}{\cancel{x-2}}$

$= -1.$

ii) R.H.L :- $\lim_{\substack{x \to 2^+ \\ x > 2}} f(x) = \lim_{x \to 2^+} \frac{|x-2|}{x-2}$

$= \lim_{x \to 2^+} \frac{\cancel{x-2}}{\cancel{x-2}}$

$= 1.$

$\therefore$ LHL $\neq$ R.H.L.

Hence The given function doesnot exist.

3) If $f: R \to R$ defined by $f(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } x = 1/2 \\ x-1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$

find $\lim_{x \to \frac{1}{2}} f(x)$ exists (or) not ?

Sol:- Given $f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \\ x-1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

i) L.H.L :- $\lim_{\substack{x \to \frac{1}{2}^- \\ x < 1/2}} f(x) = \lim_{x \to \frac{1}{2}^-} x$

$= \lim_{h \to 0} \left(\frac{1}{2} - h\right)$

$= \frac{1}{2}$

ii) R.H.L :- $\lim_{\substack{x \to \frac{1}{2}^+ \\ x > \frac{1}{2}}} f(x) = \lim_{x \to \frac{1}{2}^+} (x-1)$

$= \lim_{h \to 0} \left(\frac{1}{2} + h\right) - 1$

$= \frac{1}{2} - 1$

$= \frac{-1}{2}$

$\therefore$ L.H.L $\neq$ R.H.L.

$\therefore$ The given function does not Exists.

4) Prove That $\lim_{x \to 1} \frac{\log x}{x-1} = 1$.

Sol:- Given $f(x) = \frac{\log x}{x-1}$

W.K.T. $\log x$ $\cancel{\log(1+x)}$ $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ — (1)

$\log x = \log[(1 + \underline{(x-1)})]$

Sub $(x-1)$ value in Eq (1)

(1) $\Rightarrow (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$

Now

$$\lim_{x\to 1}\frac{\log x}{x-1} = \lim_{x\to 1}\left\{\frac{\left[(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots\right]}{x-1}\right\}$$

$$= \lim_{x\to 1}\left\{\frac{\cancel{(x-1)}\left[1-\frac{(x-1)}{2}+\frac{(x-1)^2}{3}+\cdots\right]}{\cancel{(x-1)}}\right\}$$

$$= \lim_{x\to 1}\left\{1-\frac{(x-1)}{2}+\frac{(x-1)^2}{3}-\frac{(x-1)^3}{4}+\cdots\right\}$$

$$= 1-0+0-0-\cdots$$

$$= 1$$

$$\therefore \lim_{x\to 1}\frac{\log x}{x-1} = 1.$$

Prob:-

5) prove that $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

Sol:- Given $f(x) = \frac{e^x-1}{x}$

W.K.T $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left\{\frac{\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right]-1}{x}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{\cancel{1}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots-\cancel{1}}{x}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}{x}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{\cancel{x}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)}{\cancel{x}}\right\}$$

$$= 1+\frac{0}{2!}+\frac{0}{3!}+\cdots = 1$$

$\therefore \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$

Prob:-

6) If $f: R-\{0\} \to R$ defined by $f(x) = x^2 \sin\frac{1}{x}$

prove that $\lim_{x\to 0} f(x) = 0$

sol:- Given that $f: R-\{0\} \to R$ defined by

$f(x) = x^2 \sin\frac{1}{x}$

W.K.T $-1 \leq \sin x \leq 1$

$-1 \leq \sin\frac{1}{x} \leq 1$

$-x^2 \leq x^2 . \sin\frac{1}{x} \leq x^2$

Taking limit function, we have

$\lim_{x\to 0} (-x^2) \leq \lim_{x\to 0} x^2 \sin\frac{1}{x} \leq \lim_{x\to 0} x^2$

(By Sandwich Thm Statement)

$\Rightarrow 0 \leq \lim_{x\to 0} x^2 \sin\frac{1}{x} \leq 0$

$\Rightarrow$ By Sandwich theorem. $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$

Hence. $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$

Prob:-

7) If $f(x) = \frac{e^{1/x}}{1+e^{1/x}}$ Find $\lim_{x\to 0} f(x)$ Exists or not ?

sol:- Given $f(x) = \frac{e^{1/x}}{1+e^{1/x}}$

Now L.H.L :- $\lim_{x\to 0^-} \frac{e^{1/x}}{1+e^{1/x}}$ ~~$\lim_{x\to 0} e^{1/x}$~~

$= \frac{\lim_{x\to 0^-} e^{1/x}}{1+\lim_{x\to 0^-} e^{1/x}}$

$$= \frac{0}{1+0}$$

$$= \frac{0}{1}$$

$$= 0$$

R.H.L:- $\lim_{x\to 0^+} \frac{e^{1/x}}{1+e^{1/x}}$

$$= \frac{\lim_{x\to 0} e^{1/x}}{\lim_{x\to 0}\left[e^{1/x}\left[\frac{1}{e^{1/x}}+1\right]\right]}$$

$$= \frac{\cancel{\lim_{x\to 0} e^{1/x}}}{\cancel{\lim_{x\to 0} e^{1/x}} \cdot \lim_{x\to 0}\left[\frac{1}{e^{1/x}}+1\right]}$$

$$= \frac{1}{\lim_{x\to 0}\left[e^{-1/x}+1\right]}$$

$$= \frac{1}{0+1}$$

$$= 1$$

$\therefore$ L.H.L $\neq$ R.H.L.

$\therefore$ $\lim_{x\to 0} f(x)$ does not exist.

Prob:-
(8). If $f: R-\{0\} \to R$ defined by

a) $f(x) = \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

b) $f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}$

Show that $\lim_{x\to 0} f(x)$ exists (or) does not exist

(a) Given $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

L.H.L :- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

$$= \lim_{x \to 0^-} \frac{e^{-1/x}\left[\frac{e^{1/x}}{e^{-1/x}} - 1\right]}{e^{-1/x}\left[\frac{e^{1/x}}{e^{-1/x}} + 1\right]}$$

$$= \lim_{x \to 0^-} \left[\frac{(e^{1/x})^2 - 1}{(e^{1/x})^2 + 1}\right]$$

$$= \frac{\lim_{x \to 0^-} (e^{1/x})^2 - 1}{\lim_{x \to 0^-} (e^{1/x})^2 + 1}$$

$$= \frac{0 - 1}{0 + 1} = \frac{-1}{1} = -1$$

R.H.L :- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

$$= \lim_{x \to 0^+} \frac{e^{1/x}\left[1 - \frac{e^{-1/x}}{e^{1/x}}\right]}{e^{1/x}\left[1 + \frac{e^{-1/x}}{e^{1/x}}\right]}$$

$$= \lim_{x \to 0^+} \frac{1 - (e^{-1/x})^2}{1 + (e^{-1/x})^2}$$

$$= \frac{\lim_{x \to 0^+} 1 - (e^{-1/x})^2}{\lim_{x \to 0^+} 1 + (e^{-1/x})^2} = \frac{1 - 0}{1 + 0} = \frac{1}{1} = 1$$

$\therefore$ LHL $\neq$ RHL.

$\therefore \lim_{x \to 0} f(x)$ does not exist.

b) Given $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$

L.H.L :- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{e^{1/x}-1}{e^{1/x}+1}$

$$= \frac{\lim_{x \to 0^-} \left[e^{1/x}-1\right]}{\lim_{x \to 0^-} \left[e^{1/x}+1\right]}$$

$$= \frac{0-1}{0+1}$$

$$= \frac{-1}{1} = -1$$

R.H.L :- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{e^{1/x}-1}{e^{1/x}+1}$

$$= \lim_{x \to 0^+} \frac{e^{1/x}\left[1-\frac{1}{e^{1/x}}\right]}{e^{1/x}\left[1+\frac{1}{e^{1/x}}\right]}$$

$$= \lim_{x \to 0^+} \left[\frac{1-e^{-1/x}}{1+e^{-1/x}}\right]$$

$$= \frac{\lim_{x \to 0^+} \left[1-e^{-1/x}\right]}{\lim_{x \to 0^+} \left[1+e^{-1/x}\right]}$$

$$= \frac{1-0}{1+0}$$

$$= 1$$

$\therefore$ LHL $\neq$ RHL

$\therefore \lim_{x \to 0} f(x)$ does not exists.

Continuity

Def:- Let S be a non Empty Subset of real numbers R. $f: S \to R$ is a function and $a \in S$, f is Said to be Continuous at $x = a$, if for each $\varepsilon > 0$, $\exists\ \delta > 0 \ni$, $0 < |x-a| < \delta$.

$\Rightarrow |F(x) - F(a)| < \varepsilon$

ie, $\lim\limits_{x \to a} f(x) = f(a) \ \forall a \in R$,

example:- Let f defined by $f(x) = \dfrac{x^2+x-6}{x-2}$ $\forall x \in R - \{2\}$. Find the Continuity at $x = 2$

Sol:- Given $f(x) = \dfrac{x^2+x-6}{x-2}$

Taking $x^2+x-6 = x^2+3x-2x-6$

$= x(x+3) - 2(x+3)$

$= (x-2)(x+3)$

$\therefore f(x) = \dfrac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}}$

$\therefore f(x) = (x+3)$

apply limit value

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} (x+3)$

$= 2+3$

$= \underline{5}$

$\therefore f(x)$ is Continuous at $x = 2$

1) Find the Constants a and b defined by

$$f(x) = \begin{cases} 2x+1 & \text{if } x \le 1 \\ ax^2+b & \text{if } 1<x<3 \\ 5x+2a & \text{if } x \geq 3 \end{cases}$$

is Continuous Every where.

Sol:- Since $f(x)$ is continuous at Every where

(i) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$.

$\lim_{x \to 1^-} (2x+1) = \lim_{\substack{x \to 1^+ \\ x>1}} (ax^2+b)$.

$\lim_{h \to 0} [2(1-h)+1] = \lim_{h \to 0} [a(1+h)^2+b]$

$\Rightarrow 2(1-0)+1 = a(1+0)^2+b$

$\Rightarrow 3 = a+b$ — ①

(ii) $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

$\lim_{\substack{x \to 3^- \\ x<3}} (ax^2+b) = \lim_{\substack{x \to 3^+ \\ x>3}} (5x+2a)$

$\lim_{h \to 0} (a(3-h)^2+b) = \lim_{h \to 0} (5(3+h)+2a)$

$\Rightarrow a(3-0)^2+b = 5(3+0)+2a$

$\Rightarrow 9a+b = 15+2a$

$\Rightarrow 7a+b = 15$ — ②

Solving ① and ② we get, a, b

$$\begin{array}{l} a+\cancel{b} = 3 \\ \underline{7a+\cancel{b} = 15} \\ -\quad -\quad - \\ -6a = -12 \end{array}$$

$6a = 12$

$\boxed{a = 2}$

Sub $a=2$ in ①

$2+b=3$

$b = 3-2$

$\boxed{b=1}$

2) Examine the continuity of the function 'f' defined by $f(x) = |x| + |x-1|$ at $x=0, 1$.

sol:- Given $f(x) = |x| + |x-1|$

$$|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x \leq 0 \end{cases}$$

$$|x-1| = \begin{cases} x-1 & \text{if } x-1>0 \Rightarrow x>1 \\ -(x-1) & \text{if } x-1 \leq 0 \Rightarrow x \leq 1 \end{cases}$$

For $x \leq 0$ if $f(x) = -x-x+1 = 1-2x$

$0<x<1$ if $f(x) = x-x+1 = 1$

$x \geq 1$ if $f(x) = x+x-1 = 2x-1$

i) Continuity at $x=0$:-

At $x=0 \Rightarrow f(0) = 1-2(0)$

$= 1$

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0^- \\ x<0}} 1-2x$$

$$= 1-2(0)$$

$$= 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0^+ \\ x>0}} 1$$

$$= 1$$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 1$

$\therefore$ f is continuous at $x=0$.

(ii) Continuity at $x=1$

at $x=1$, $\Rightarrow f(x) = 2x-1$

$$f(1) = 1$$

Now $\lim_{\substack{x\to 1^- \\ x<1}} f(x) = \lim_{\substack{x\to 1^- \\ x<1}} (1)$

$$= 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{\substack{x\to 1 \\ x>1}} 2x-1$$

$$= 2(1)-1$$

$$= 1$$

$\therefore \lim_{x\to 9^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 1$

$\therefore$ f is continuous at $x=1$.

3) Uniformly Continuity :-

Let $f: S \to R$ be a function then f is said to Uniformly Continuous on S. if, given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $x_1, x_2 \in S$,

$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

Prob:- Show that f is defined by $f(x) = x^3$ is Uniformly Continuous on $[-2, 2]$

Sol:- Given that $f(x) = x^3$

Let $x_1, x_2 \in [-2, 2]$ and $\varepsilon > 0$,

$\Rightarrow |x_1| \leq 2, \quad |x_2| \leq 2$

Consider $|f(x_1) - f(x_2)| = |x_1^3 - x_2^3|$

$= |(x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2)|$

$\leq |x_1 - x_2| \, |x_1^2 + x_1 x_2 + x_2^2|$

$\leq |x_1 - x_2| \left[|x_1|^2 + |x_1 x_2| + |x_2|^2 \right]$

$\leq |x_1 - x_2| \left[2^2 + 2 \cdot 2 + 2^2 \right]$

$\leq |x_1 - x_2| \left[4 + 4 + 4 \right]$

$\leq |x_1 - x_2| \cdot 12$

$\therefore |f(x_1) - f(x_2)| \leq 12 \cdot |x_1 - x_2|$

$\therefore |f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

where $\delta = \frac{\varepsilon}{12}$, Hence $f(x)$ is Uniformly Continuous on $[-2, 2]$

4) Prob:- Show that f is defined by $f(x) = \sqrt{x}$ is Uniformly Continuous on $[0,1]$.

Sol:- Given that $f(x) = \sqrt{x}$

Let $x_1, x_2 \in [0,1]$ & $\varepsilon > 0$

$\Rightarrow |x_1| \leq 0$

$\Rightarrow |x_2| \leq 1$

Consider $|f(x_1) - f(x_2)| = |\sqrt{x_1} - \sqrt{x_2}|$

$$= \left| \sqrt{x_1} - \sqrt{x_2} \times \frac{\sqrt{x_1} + \sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \right|$$

$$= \left| \frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}} \right|$$

$$\leq \frac{|x_1 - x_2|}{|\sqrt{x_1} + \sqrt{x_2}|}$$

$$\leq \frac{|x_1 - x_2|}{|0 + 1|}$$

$$\leq |x_1 - x_2|$$

$\therefore$ By definition of Uniform Continuity

$|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

where $\delta = \varepsilon > 0$

$\therefore f(x)$ is Uniformly continuous on $[0,1]$.

5) Show that f is defined by $f(x) = x^2+2x+2$. is Uniformly Continuous on $[1,2]$

Sol:- Given that $f(x) = x^2+2x+2$

Let $x_1, x_2 \in [1,2]$ and $\varepsilon > 0$.

$\Rightarrow |x_1| \leq 1$

$\Rightarrow |x_2| \leq 2$

Consider $|f(x_1) - f(x_2)|$

$$= |(x_1^2+2x_1+2) - (x_2^2+2x_2+2)|$$

$$= |x_1^2+2x_1+\cancel{2} - x_2^2 - 2x_2 - \cancel{2}|$$

$$= |x_1^2+2x_1 - x_2^2 - 2x_2|$$

$$= |x_1^2 - x_2^2 + 2x_1 - 2x_2|$$

$$= |x_1^2 - x_2^2 + 2(x_1 - x_2)|$$

$$= |(x_1-x_2)(x_1+x_2) + 2(x_1-x_2)|$$

$$= |(x_1-x_2)[x_1+x_2+2]|$$

$$\leq |x_1-x_2| [|x_1|+|x_2|+|2|]$$

$$\leq |x_1-x_2| [1+2+2]$$

$$\leq |x_1-x_2| 5$$

$\therefore |f(x_1) - f(x_2)| \leq 5|x_1-x_2|$

$\therefore |f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1-x_2| < \delta$

where $\delta = \dfrac{\varepsilon}{5}$

$\therefore f(x)$ is Uniformly continuous on $[1,2]$

Def:- Partition of a closed interval:-

→ Let $[a,b]$ be a closed interval. Let $a = x_0 < x_1 < x_2 \cdots$
$\cdots < x_n = b$ then the finite set $P = \{x_0, x_1, \cdots x_n\}$
is called the partitions of $[a,b]$. Then $n+1$ points
are called partition points of P.

→ $[x_0, x_1], [x_1, x_2], \cdots [x_{r-1}, x_r] \cdots [x_{n-1}, x_n]$
are called n subintervals of partition P.

→ The r^{th} subinterval is denoted by I_r and its length
is defined as length $= x_r - x_{r-1}$ and it is denoted by
δ_r.

1) **Theorem:-** If $f : [a,b] \to R$ is continuous on $[a,b]$.
Then f is bounded on $[a,b]$.

Proof:- Given $f : [a,b] \to R$. is continuous
on $[a,b]$, we can divide $[a,b]$ into finite number
of subinterval. Say $[a = a_0, a_1], [a_1, a_2], \cdots$
$\cdots [a_{n-1}, a_n = b]$.

$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ — ① whenever x_1, x_2 belongs
to the same subinterval.

Let x be any point in the first sub interval
$[a, a_1]$ then from eq ① $|f(x) - f(a_1)| < \varepsilon$.,
whenever $x \in [a, a_1]$

$$|f(x)| = |f(x) - f(a) + f(a)|$$
$$\leq |f(x) - f(a)| + |f(a)|$$
$$|f(x)| < \varepsilon + |f(a)|$$

In Particular if $x = a$, then $|f(a_1)| < \varepsilon + |f(a)|$ — (2)

Let x be any point in the second sub interval then from eq (1)

$$|f(x) - f(a_1)| < \varepsilon \quad \forall x \in [a_1, a_2]$$

$$|f(x)| = |f(x) - f(a_1) + f(a_1)|$$
$$\leq |f(x) - f(a_1)| + |f(a_1)|$$

$$|f|f(x)| < \varepsilon + |f(a_1)|$$
$$< \varepsilon + \varepsilon + |f(a)| \text{ (from (2))}$$

$$|f(x)| < 2\varepsilon + |f(a)|.$$

In Particular if $x = a_2$ then $|f(a_2)| < 2\varepsilon + |f(a)|$

Proceeding in this way we get $|f(x)| < n\varepsilon + |f(a)|$ $\forall x \in [a_{n-1}, a_n]$.

∴ The inequality holds for the whole interval, that is $\forall x \in [a, b]$

$$\therefore |f(x) - f(a)| < |n\varepsilon + f(a) - f(a)|$$

$$|f(x) - f(a)| < n\varepsilon$$

$$\therefore -n\varepsilon < f(x) - f(a) < n\varepsilon$$
$$< n\varepsilon + f(a).$$

$$\Rightarrow f(a) - n\varepsilon < f(x)$$

∴ f is bounded.

<u>example:-</u> Every Continuous function is bounded. But bounded function neednot be continuous.

<u>Sol:-</u> $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$

$\therefore \quad -1 \leq f(x) \leq 1$

$\therefore$ f is bounded on $\mathbb{R}$.

Hence Every continuous function f is bounded.

conversely, we have to prove that bounded function neednot be continuous,

Now,

L.H.L:- $\lim_{\substack{x \to 0^- \\ x < 0}} f(x) = \lim_{x \to 0^-} (-1) = -1$

R.H.L:- $\lim_{\substack{x \to 0^+ \\ x > 0}} f(x) = \lim_{x \to 0^+} (1) = 1$

$\therefore$ LHL $\neq$ RHL

$\therefore$ f is not continuous at $x = 0$.

Hence. Every bounded function need not be continuous.

<u>Def:-</u>